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Tree Graph Inequalities and Critical Behavior in Percolation Models

Michael Aizenman[1,3,5] **and Charles M. Newman**[2,4,6]

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Various inequalities are derived and used for the study of the critical behavior in independent percolation models. In particular, we consider the critical exponent γ associated with the expected cluster size χ, and the structure of the n-site connection probabilities $\tau = \tau_n(x_1, \ldots, x_n)$. It is shown that quite generally $\gamma \geqslant 1$. The upper critical dimension, above which γ attains the Bethe lattice value 1, is characterized both in terms of the geometry of incipient clusters and a diagramatic convergence condition. For homogeneous d-dimensional lattices with $\tau(x, y) = O(|x-y|^{-(d-2+\eta)})$, at $p = p_c$, our criterion shows that $\gamma = 1$ if $\eta > (6-d)/3$. The connectivity functions τ_n are generally bounded by tree diagrams which involve the two-point function. We conjecture that above the critical dimension the asymptotic behavior of τ_n, in the critical regime, is actually given by such tree diagrams modified by a nonsingular vertex factor. Other results deal with the exponential decay of the cluster-size distribution and the function $\tau_2(x, y)$.

KEY WORDS: Percolation; critical exponents; correlation functions; connectivity inequalities; upper critical dimension; cluster size distribution; rigorous results.

1. INTRODUCTION

Percolation is the phenomenon of formation of infinite connected clusters in a system of random geometric objects, which may, for example, be the

[1] Departments of Mathematics and Physics, Rutgers University, New Brunswick, New Jersey 08903.

[2] Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

[3] A. P. Sloan Foundation Research Fellow. Research supported in part by the National Science Foundation Grant No. PHY-8301493.

[4] Research supported in part by the National Science Foundation Grant No. MCS80-19384.

[5] Visiting Address: Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot, Israel.

[6] Visiting Address: Institute of Mathematics and Computer Science, The Hebrew University, Jerusalem, Israel.

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set of "conducting" lattice bonds, or the set of "occupied" sites on a regular lattice. Even if the basic local variables are non-interacting, i.e., are independently distributed, globally the system may exhibit a transition (as a density parameter p is varied) from a nonpercolating phase to a percolating phase.

As with other phase transitions, one expects the critical behavior to be mostly affected by the dimension of the lattice. It is well understood that (for finite-range systems) there is a *lower critical dimension*, here $d = 1$, only above which the phase transition occurs at a regular value of p (that is, $p \neq 1$). Furthermore, it is expected that there is also an *upper critical dimension*, above which the critical behavior takes a very simple form. The renormalization group approach offers an appealing picture of this behavior in which it is argued that the fixed point (in some very large space) towards which a critical system is driven by scaling is rather simple in high dimensions. It has been argued that for percolation the upper critical dimension is $d = 6$. However, so far very little has been achieved in the direction of providing rigorous arguments to support, and explain, these predictions. The results which are presented here offer a step in this direction.

Our analysis is inspired by the arguments which were developed in Ref. 1, where a similar problem was solved for Ising systems and ϕ^4 fields (where the upper critical dimension is 4). We have been informed that some of the inequalities which are derived and used here have been also found by J. Fröhlich,[2] whose work[3] provides a somewhat parallel analysis to Ref. 1. Some of our results are reviewed in Ref. 4.

The basic quantities which we consider are the n-site connectivity functions $\tau(x_1, \ldots, x_n)$ $(= \tau_n)$, which are defined (in Sections 2 and 4) as the probabilities that the n sites are all connected. The sum

$$\chi = \sum_x \tau(0, x) \tag{1.1}$$

is the expected value of the *size* of the cluster, $C(0)$, of sites which are connected to the origin 0. Higher moments of the cluster size, $|C(0)|$, are similarly expressed as sums (with one site fixed) of τ_n's, for $n > 2$.

The quantity χ, which is a monotone increasing function of the density p of the connecting bonds, diverges at a critical value p_c (denoted as p_T and π_c in Refs. 5 and 13). We present some general results about the critical behavior of χ, and discuss the critical exponent γ defined by

$$\chi \cong (p_c - p)^{-\gamma} \quad \text{(in some appropriate sense)}, \qquad \text{as} \quad p \nearrow p_c \tag{1.2}$$

Our main results fall into two classes:

(i) Bounds on various quantities of interest (including p_c, γ, χ, τ_n and the correlation length ξ) which hold for general homogeneous lattices. Some of these are essentially satisfied as equalities for approximating Bethe lattice models.

(ii) Some heuristic geometric ideas and a rigorous diagrammatic convergence criterion for the upper critical dimension. Above this dimension the critical behavior of both χ and the functions τ_n simplifies considerably; $\gamma = 1$ and we conjecture that the structure of τ_n is actually well described by the bounds mentioned above (up to corrections by factors which are regular at p_c).

Most of the results apply to general homogeneous (i.e., translation invariant) percolation models, in which various bonds may be occupied, or connecting, with probabilities which, for convenience, depend monotonically on a single parameter β. This general setup is introduced in Section 2. Some of the *main results* are listed below.

1. A proof that in the general case of homogeneous, independent, bond percolation models the cluster size, $\chi(\beta)$, actually diverges as $\beta \nearrow \beta_c$ (Section 3). For finite-range models this also implies that $\lim_{\beta \nearrow \beta_c} \xi(\beta) = \infty$ (ξ the correlation length).

For finite-range models this result could have also been proven by means of an inequality (see Section 5.2) fashion after the Simon–Lieb inequality[6,7] for ferromagnetic spin systems. However, the argument presented here offers a simpler treatment, which can be applied to those systems as well. Furthermore, this "continuity" of χ holds even for long-range models, in which the infinite cluster density may be discontinuous.[8]

2. A proof that the Bethe lattice approximation (described in Section 2.2) provides, quite generally, bounds not only for the critical density (which was known) but also for the critical exponent γ. Specifically,

$$\beta_c \geqslant \beta_c^{\mathrm{B.L.}} \qquad (\text{or } p_c \geqslant p_c^{\mathrm{B.L.}} \text{ for models with a single density } p) \tag{1.3}$$

and

$$\gamma \geqslant 1 \qquad (= \gamma^{\mathrm{B.L.}}) \tag{1.4}$$

Both statements follow from a single upper bound on $|(d/d\beta)\chi(\beta)^{-1}|$ [or $|(d/dp)\chi(p)^{-1}|$, where appropriate], which by integration yields upper and lower bounds on χ (Section 3). For example, for the standard model on $\mathbb{Z}^d$, with only the nearest-neighbor bonds being occupied with probability p, the resulting bounds are:

$$\left[2d(p_c - p)\right]^{-1} \leqslant \chi(p) \leqslant \left|2d\left(\frac{1}{2d} - p\right)\right|_{+}^{-1} \tag{1.5}$$

[For a related quantity $\hat{\chi}$ ($\leqslant \chi$) one can replace $(2d)$ in (1.5) by $(2d-1)$, which is associated with a slightly better Bethe lattice approximation.]

3. Upper bounds for the connectivity functions τ_n in terms of functions of lower order (Section 4). The ultimate reduction states that $\tau_n(x_1, \ldots, x_n)$ is bounded by the sum of products of the two-point function which correspond to all the tree diagrams having $x_1, \ldots, x_n$ as external vertices, and valence 3 at the internal vertices. The simplest such bound is

$$\tau_3(x_1, x_2, x_3) \leqslant \sum_y \tau(x_1, y)\tau(x_2, y)\tau(x_3, y) \tag{1.6}$$

(for *any* independent bond percolation model).

4. General exponential bounds on the cluster size distribution, for any $\beta < \beta_c$ (of $p < p_c$). Specifically,

$$\text{Prob}(|C(0)| \geqslant k) \leqslant (e/k)^{1/2} e^{-k/(2\chi^2)} \tag{1.7}$$

for $k \geqslant \chi^2$ (Section 5.1). The derivation of and the constants in (1.7) improve previous results of Kesten.[13] Moreover the bound (1.7) holds even for long-range percolation models, including those for which $\tau(x, y)$ does not decay exponentially.

5. Exponential bounds for the two-point function for finite-range models, in which the connecting bonds are of bounded length. For the standard nearest-neighbor model on $\mathbb{Z}^d$ we get

$$\tau(x, y) \leqslant (1 - \chi^{-1})^{\|x-y\|} \leqslant e^{-\|x-y\|/\chi} \tag{1.8}$$

where $\|x\| = \sum_{i=1}^d |x_i|$.

For the proof of (1.8) we derive an analog of the Simon inequality (with Leib's improvement). The validity of this inequality for percolation models has been realized by a number of people—see Section 5.2. However, the bound (1.8) represents a slight improvement in the application of such inequalities.

6. A criterion for the upper-critical dimension. An explicit formula for $|d\chi^{-1}/dp|$ (Section 3.2) shows that the critical behavior of χ simplifies considerably in dimensions in which the probability of neighboring "incipient clusters" to intersect is less than one (uniformly as $p \nearrow p_c$). In such models the critical exponent γ attains the Bethe-lattice value:

$$\gamma = 1 \qquad (= \gamma^{\text{B.L.}}) \tag{1.9}$$

in the strong sense that $\chi(p)$ is bounded both above and below by expressions of the form const $|p_c - p|_+^{-1}$. As a concrete criterion, we prove (in Section 6) that in finite-range models (1.9) is indeed satisfied if the "triangle diagram":

$$\nabla = \sum_{x,y} \tau(0,x)\tau(x,y)\tau(y,0) \tag{1.10}$$

is *finite* at $p = p_c$ (or, uniformly bounded for $p < p_c$).

The above criterion is reminiscent of an analogous yet significantly different statement which holds for the magnetic susceptibility in a class of ferromagnetic spin systems. In the latter case, the sufficiency criterion for the analog of (1.9) (derived in Ref. 1, and extended to the critical dimension $d = 4$ in Ref. 9) is the finiteness at $\beta = \beta_c$ of the bubble diagram

$$B = \sum_{x} S(0,x)S(x,0) \tag{1.11}$$

where $S(0,x)$ is the spin correlation function.

For a simple comparison of the two criteria, let us rewrite the quantities, for the case of a cubic lattice $\mathbb{Z}^d$, in terms of the Fourier transform:

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(0,x)e^{i(k,x)} \tag{1.12}$$

One gets

$$\nabla = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} dk\, \hat{\tau}(k)^3 \tag{1.13}$$

while

$$B = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} dk\, \hat{S}(k)^2 \tag{1.14}$$

It is known (by the "reflection positivity" argument of Ref. 10) that the spin correlation function, in ferromagnetic spin systems with nearest-neighbor interactions, satisfies

$$(0 \leqslant) \quad \hat{S}(k) \leqslant \left[\beta \sum_{i=1}^{d} \left(2 \sin \frac{k_i}{2} \right)^2 \right]^{-1} \qquad \left(\cong \frac{1}{\beta k^2} \text{ for } k \ll 1 \right) \tag{1.15}$$

for all $\beta < \beta_c$. Thus the above criterion with B is met in dimensions $d > 4$. Were the analog of (1.15) to hold for $\hat{\tau}(k)$, our criterion would show that $\gamma = 1$ in any dimension above $d = 6$.

One may find in the above results a tenuous support for the notion that the upper critical dimension for percolation is $d = 6$.[11] However, the analog of (1.15) is expected to be invalid for $\hat{\tau}(k)$ in some dimensions below 6.[12] Thus, our results only prove that $\gamma = 1$ in a particular dimension d if the critical exponent η, defined by: $\hat{\tau}(k) \simeq \text{const}/k^{2-\eta}$ (at $p = p_c$), satisfies:

$$\eta > (6 - d)/3 \tag{1.16}$$

7. We expect the simplification in the critical behavior, above the upper critical dimension to show not only in the critical exponents, but also in the structure of the connectivity functions τ_n. For reasons mentioned in Section 4.1 we conjecture there that the latter reduce, asymptotically, to combinations of the two-point function—given by the tree diagrams, with some nonsingular vertex factor $0 < G < 1$.

Other technically useful results not mentioned above, include a general positivity statement for $\hat{\tau}(k)$ (Section 3.2) and the difference inequalities for τ_n of Sections 4.2 and 5.2.

In most of the paper we refer to bond percolation models. However, the analysis has a natural extension to site percolation, which is briefly reviewed in Section 7.

2. BOND PERCOLATION MODELS

2.1. The Model

We consider here quite general independent (Bernoulli) bond percolation models, paying special attention to systems on homogeneous (i.e., translation invariant) lattices. (Site percolation models are discussed in Section 7.)

The lattice is a countable set of sites, denoted by $\mathbb{L}$, with a group of isomorphisms (translations $T: \mathbb{L} \to \mathbb{L}$) which acts transitively on $\mathbb{L}$. We refer to pairs of sites as bonds, $b = \{x, y\}$, and assign to each bond a random variable, $n_b = 0$ or 1. The variables $\{n_b\}$ are jointly independent, with the probabilities

$$\mathrm{Prob}(n_b = 1) = K_b(\beta) \tag{2.1}$$

which depend on the parameter $\beta \in [0, \infty)$, and have the properties listed below.

(i) *Homogeneity* (when stated):

$$K_{\{Tx,Ty\}}(\beta) = K_{\{x,y\}}(\beta) \tag{2.2}$$

(ii) The functions $K_b(\beta)$ are *nondecreasing* in β, and *locally summable* in b, i.e.,

$$\sup_{x \in \mathbb{L}} \bar{K}_x(\beta) < \infty \tag{2.3}$$

where

$$\bar{K}_x(\beta) = \sum_{b \ni x} K_b(\beta) \tag{2.4}$$

(In the homogeneous case, the subscript x in $\bar{K}_x$ will often be omitted.)

(iii) With no further loss of generality, we also assume that $K_b(\beta)$ and $\bar{K}_x(\beta)$ are *differentiable* functions with

$$K_b(0) = 0 \qquad \text{for all the bonds } b \tag{2.5}$$

and

$$\frac{d\bar{K}_x(\beta)}{d\beta} = \sum_{b \ni x} \frac{dK_b(\beta)}{d\beta} \leqslant C < \infty \tag{2.6}$$

for all $\beta \geqslant 0$, and $x \in \mathbb{L}$.

While the following condition will not be used, it may also be assumed that

$$\sup_b \lim_{\beta \to \infty} K_b(\beta) = 1$$

For a given configuration of values of $\{n_b\}$, we regard each bond with $n_b = 1$ as occupied, or connecting, and partition the lattice into connected components.

An important, and standard, example is $\mathbb{L} = \mathbb{Z}^d$ (the d-dimensional cubic lattice) with only the nearest-neighbor connections, i.e.,

$$K_{\{x,y\}}(\beta) = \begin{cases} p(\beta), & |x-y| = 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

The natural parameter for such models is, of course, $p \in [0, 1]$ itself.

Denoting by $C(x)$ the (n-dependent) connected cluster containing the site $x \in \mathbb{L}$, we define

$$\tau(x, y) = \operatorname{Prob}(y \in C(x)) \qquad [\equiv \operatorname{Prob}(C(x) = C(y))] \tag{2.8}$$

and

$$\chi = \sum_{x \in \mathbb{L}} \tau(0, x) \tag{2.9}$$

Thus $\tau(x, y)$ is the probability that x and y are connected. Furthermore, $\chi = \chi(\beta)$ is the expectation value of the cluster size:

$$\chi = \langle |C(0)| \rangle \tag{2.10}$$

where $|C|$ is the number of points in the cluster.

Remark. In order to explain the relation (2.10) it is useful to introduce the indicator functions (of $\{n_b\}$)

$$I[x \in C(0)] = \begin{cases} 1, & \text{if } x \in C(0) \\ 0, & \text{if } x \notin C(0) \end{cases} \tag{2.11}$$

One gets

$$\tau(0, x) = \langle I[x \in C(0)] \rangle \tag{2.12}$$

and thus

$$\chi = \sum_x \langle I[x \in C(0)] \rangle = \left\langle \sum_x I[x \in C(0)] \right\rangle = \langle |C(0)| \rangle \qquad (2.13)$$

It is well known that the nearest-neighbor models on $\mathbb{Z}^d$, $d \geqslant 2$ (and models dominating those) exhibit a phase transition, at which $\chi(\beta)$ (which is a nondecreasing function of β) diverges.

One of the main questions addressed in this paper is the value of the critical exponent

$$\gamma = -\liminf_{\beta \nearrow \beta_c} \frac{\log \chi(\beta)}{\log(\beta_c - \beta)} \qquad (2.14)$$

which characterizes the critical behavior ($\chi(\beta) \cong (\beta_c - \beta)^{-\gamma}$) in the vicinity of the critical point:

$$\beta_c = \sup\{\beta \mid \chi(\beta) < \infty\} \qquad (2.15)$$

Remark. $p_c \stackrel{\text{def}}{=} p(\beta_c)$ is p_T, or π_c, in the notation of Refs. 5 and 13. It has not yet been rigorously proven, except for $d = 2$, that β_c is also the percolation threshold, where infinite clusters first appear.

2.2. The Bethe Lattice Approximation

When pressed for a quick estimate of p_c and the critical exponent γ, one is tempted to reduce the complexity of the problem and consider an analogous model on a Bethe lattice. While this is a very simplistic treatment of a nontrivial effect, we shall demonstrate that the values which it yields form, in fact, rigorous lower bounds.

In this "approximation," the nearest-neighbor bond percolation model on $\mathbb{Z}^d$ is replaced by percolation on the Cayley tree on which each site has $2d$ neighbors. The probability that a site x is connected to 0 is $p^{\text{dist}(0,x)}$. The removal of one of the bonds which connect to 0 splits the tree into the "top" part, which may still be connected to 0, and the "root system." Denoting by χ^T the quantity χ defined for the top only, one readily obtains

$$\chi^T = \sum_{k=0}^{\infty} (2d-1)^k p^k \qquad (2.16)$$

and

$$\chi = (1+p)\chi^T \qquad (2.17)$$

Thus,

$$\chi(p) = \frac{1+p}{(2d-1)} \frac{1}{|p_c^{\text{B.L.}} - p|_+^{\gamma^{\text{B.L.}}}} \tag{2.18}$$

with the critical probability, and the critical exponent

$$p_c^{\text{B.L.}} = (2d-1)^{-1}, \qquad \gamma^{\text{B.L.}} = 1 \tag{2.19}$$

On the tree used in the above approximation, the set of links connected to each vertex is isomorphic to, and can be labeled by, the set $B = \{b \ni 0 \mid K_b(\cdot) \not\equiv 0\}$ of the relevant bonds of the lattice $\mathbb{L}$ which contain 0. Such a construction can be extended to the more general case, described by (2.1). However, a mild complication arises from the fact that the links along any "ascending path" are constrained not to have certain pairs of labels in succession. The calculation is less tedious if one replaces this tree with one where the constraint is removed. In this approximation, which we denote by an asterisk, one gets

$$\chi^*(\beta) = \left[1 - \overline{K}(\beta)\right]^{-1} \tag{2.20}$$

which diverges at β_c^*, where $\sum_{b \ni 0} K_b(\beta_c^*) = 1$. Furthermore, for $\beta \cong \beta_c^*$,

$$\chi^*(\beta) \cong \left(\frac{d}{d\beta} \overline{K}(\beta) \Big|_{\beta_c^*} \right)^{-1} \frac{1}{|\beta_c^* - \beta|_+^1} \tag{2.21}$$

i.e., the critical exponent γ is still 1.

In the next section we prove, by means of a single differential inequality, that the above simple calculation leads to rigorous bounds for both β_c and the critical exponent γ. However, since the first statement is in fact an older, and commonly made observation, let us present it here.

Proposition 2.1. The critical point [defined by (2.15)] for the bond percolation model (2.1) satisfies

$$\beta_c \geqslant \beta_c^* \tag{2.22}$$

Furthermore, for the nearest-neighbor model on $\mathbb{Z}^d$,

$$p_c \geqslant \frac{1}{2d-1} \tag{2.23}$$

[which is an improvement since $p(\beta_c^*) = 1/2d$].

Proof. If $x \in \mathbb{Z}^d$ is connected to 0 then there is a self-avoiding path (along the bonds of the lattice) which connects the two sites, all of whose bonds are occupied. The probability of such an event equals the probability that the site on the tree described above, which corresponds to the given path, is connected there to the origin. The summation over such events leads to the upper bounds

$$\chi(\beta) \leqslant \chi^{\text{B.L.}}(\beta) \leqslant \chi^*(\beta) \tag{2.24}$$

which imply the corresponding inequalities among the critical points. ■

Remarks. (i) It is clear from the above argument that the bounds (2.22)–(2.24) can be improved, by a better count of the self-avoiding paths. An estimate of this type was used by M. Fisher, in his derivation of the mean-field upper bound for the critical temperature in Ising models.(14)

(ii) It might be pointed out that the bound (2.24) does not provide us with any information on the critical exponent γ, since actually $\beta_c \neq \beta_c^{\text{B.L.}}$. Nevertheless, we shall next prove that the Bethe lattice value of γ is in fact a lower bound.

3. RIGOROUS RESULTS ON THE CRITICAL BEHAVIOR OF χ

An interesting feature of the Bethe lattice approximation is that $\gamma^{\text{B.L.}}$ has the "universal" value 1, which is independent of the details of the model, including the value of d. The numerical evidence is that in low dimensions (e.g., $d = 2$) this value is not correct. However it is expected that a strict equality $\gamma = \gamma^{\text{B.L.}}$ holds above an upper critical dimension ($d = 6$?). We shall now prove that 1 is in general a lower bound for γ, and derive a criterion for the upper critical dimension.

3.1. A Lower Bound for γ

Proposition 3.1. In any homogeneous (independent) bond percolation model the critical exponent γ, defined by (2.14), satisfies

$$\gamma \geqslant 1 \tag{3.1}$$

Furthermore, for $\beta < \beta_c$,

$$\chi(\beta) \geqslant \left[\bar{K}(\beta_c) - \bar{K}(\beta)\right]^{-1} \tag{3.2}$$

The above bounds are derived from the following differential inequality for $\chi(\beta)^{-1}$.

Lemma 3.1. The quantity $\chi(\beta)^{-1}$ is continuous at β_c [i.e., $\lim_{\beta \uparrow \beta_c} \chi(\beta) = \infty$], and satisfies

$$(0 \leqslant) \quad -\frac{d}{d\beta}\chi(\beta)^{-1} \leqslant \frac{d}{d\beta}\bar{K}(\beta) \qquad \text{for} \quad \beta < \beta_c \tag{3.3}$$

[The derivative $(d/d\beta)\chi^{-1}$ is interpreted here in the weak sense. The results of Section 5.1 imply that for the nearest-neighbor model on $\mathbb{Z}^d$, $\chi(p)$ is in fact real analytic for $p \in (0, p_c)$.]

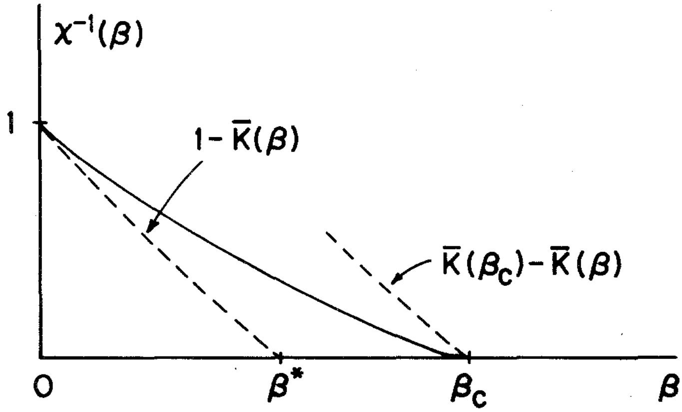

Fig. 1. Schematic graph of the function $\chi(\beta)^{-1}$. The finiteness of the slope implies $\gamma \geqslant 1$. $\gamma > 1$ is possible only if the slope vanishes at β_c. That has an implication about the geometry of the incipient cluster, which is discussed in Section 3.2.

Before proving the lemma, let us present its application.

Proof of Proposition 3.1. The boundary values of $\chi(\beta)^{-1}$ in the interval $[0, \beta_c]$ are

$$\chi(0)^{-1} = 1 \quad \text{and} \quad \chi(\beta_c)^{-1} = 0 \tag{3.4}$$

Thus, the integration of (3.3) from the two ends of this interval (see Fig. 1) yields (for $\beta \in [0, \beta_c)$)

$$1 - \int_0^{\beta} \frac{d}{ds} \bar{K}(s)\, ds \leqslant \chi(\beta)^{-1} \leqslant 0 + \int_{\beta}^{\beta_c} \frac{d}{ds} \bar{K}(s)\, ds$$

or

$$1 - \bar{K}(\beta) \leqslant \chi(\beta)^{-1} \leqslant \bar{K}(\beta_c) - \bar{K}(\beta) \tag{3.5}$$

Notice that in addition to implying the claimed (3.2), (3.5) includes also the $\chi^*(\beta)$ bound of (2.24) and hence the inequality $\beta_c \geqslant \beta_c^*$.

The bound on γ in (3.1) follows from (3.2) and (2.4). ■

For the nearest neighbor model on $\mathbb{Z}^d$, with $\bar{K} = 2dp$, the bound (3.3) shows that $|d\chi^{-1}/dp| \leqslant 2d$. Omitting the proofs, let us remark that by considering a quantity which is an analog of χ^T, of (2.16), one can produce bounds which are more reminiscent of the *first* of the two Bethe lattice approximations discussed above. Defining $\hat{\chi}$ as $\sum_x \tau(0,x)$ in a system in which one of the bonds containing 0 has been "removed," one can show (by combining the arguments used next with some of the ideas used in section 4.2) that

$$\hat{\chi} \leqslant \chi \leqslant (1+p)\hat{\chi} \tag{3.6}$$

and

$$\left| \frac{d\hat{\chi}^{-1}}{dp} \right| \leqslant 2d - 1 \tag{3.7}$$

These bounds are quite reminiscent of (2.16)–(2.18), and imply

$$\chi(p) \geqslant \hat{\chi}(p) \geqslant \left[(2d-1)|p_c - p|_+ \right]^{-1} \tag{3.8}$$

which is a slight improvement over (3.2).

To prove Lemma 3.1 we shall first derive the following result for finite systems—without assuming the homogeneity condition (i) of Section 2.

Lemma 3.2. In a percolation model on a finite set $\mathbb{L}$,

$$\frac{d}{d\beta}\tau(x, y) \leqslant \sum_{u,v \in \mathbb{L}} \tau(x,u)\left(dK_{\{u,v\}}/d\beta\right)\tau(v, y) \tag{3.9}$$

Proof. For a given configuration of $\{n_b\}$ (i.e., of connecting bonds), we say that the bond $\{u,v\}$ is *pivotal* for the connection of x with y if the two points are connected in the configuration which is obtained from $\{n_b\}$ by setting $n_{\{u,v\}} = 1$, and are disconnected in the configuration obtained by setting $n_{\{u,v\}} = 0$. By Russo's formula (or a simple direct argument)

$$\frac{\partial}{\partial K_{\{u,v\}}}\tau(x, y) = \mathrm{Prob}(\{u,v\} \text{ is pivotal for the connection of } x \text{ with } y) \tag{3.10}$$

[where we view $\tau(x, y)$ as a function of $\{K_b\}$].

We shall denote now by $\tilde{C}^{\{u,v\}}(z)$, or just $\tilde{C}(z)$, the cluster of sites which (in a given configuration $\{n_b\}$) are connected to z—even after $n_{\{u,v\}}$ is set to 0. Reexpressing the right-hand side of (3.10), we get

$$\begin{aligned}\frac{\partial}{\partial K_{\{u,v\}}}\tau(x, y) &= \mathrm{Prob}(x \in \tilde{C}(u), y \in \tilde{C}(v) \text{ and } \tilde{C}(u) \cap \tilde{C}(v) = \varnothing)\\ &\quad + a(u \leftrightarrow v) \text{ permutation of the above}\\ &= \mathrm{Prob}(x \in \tilde{C}(u), v \notin \tilde{C}(u))\\ &\quad \times \mathrm{Prob}(\tilde{C}(v) \ni y \mid x \in, v \notin \tilde{C}(u))\\ &\quad + a(u \leftrightarrow v) \text{ permutation}\end{aligned} \tag{3.11}$$

where the last factor is a *conditional probability*.

The first factor in the right-hand side of (3.11) is clearly bounded by $\tau(x,u)$. Furthermore, the second factor is bounded by $\tau(v, y)$—since the specification that $\tilde{C}^{\{u,v\}}(u) = A$, from some $A \subset \mathbb{L}$, does not affect (in the independent model) the distribution of the bond variables of $\mathbb{L}\backslash A$. (This point is made more explicit in Section 4.2.)

Hence

$$\frac{\partial}{\partial K_{\{u,v\}}} \tau(x,y) \leqslant \tau(x,u)\tau(v,y) + \tau(x,v)\tau(u,y) \tag{3.12}$$

and therefore

$$\begin{aligned} \frac{d}{d\beta}\tau(x,y) &= \frac{1}{2}\sum_{u,v\in\mathbb{L}}\left[\frac{d}{d\beta}K_{\{u,v\}}(\beta)\right]\frac{\partial}{\partial K_{\{u,v\}}}\tau(x,y) \\ &\leqslant \sum_{u,v\in\mathbb{L}}\tau(x,u)\frac{d}{d\beta}K_{\{u,v\}}(\beta)\tau(v,y) \quad \blacksquare \end{aligned} \tag{3.13}$$

We shall now use (3.9) to prove the main lemma, which deals with homogeneous (infinite) systems.

Proof of Lemma 3.1. Since $\chi = \sum_x \tau(0,x)$, the bound (3.3) may be derived directly from (3.9). However, this argument still leaves the possibility of a *jump discontinuity* of $\chi(\beta)^{-1}$ at β_c. [$\chi(\beta)^{-1} \equiv 0$ in (β_c, ∞), by the definition of β_c and the monotonicity of $\chi(\beta)$.] We shall therefore be more careful.

Let $\{0\} \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \mathbb{L}$ be a sequence of finite subsets of $\mathbb{L}$, with $\bigcup_{n=0}^{\infty}\Lambda_n = \mathbb{L}$. Denoting by $\tau^n(x,y)$ the probability that x and y are connected *in* Λ_n (i.e., by the occupied bonds whose both ends are in Λ_n), we define

$$\hat{\chi}_n = \sup_{x\in\Lambda_n}\sum_{y\in\Lambda_n}\tau^n(x,y) \tag{3.14}$$

Clearly

$$\chi(\beta) \geqslant \hat{\chi}_n(\beta) \geqslant \sum_{y\in\Lambda_n}\tau^n(0,y) \tag{3.15}$$

By the bounded convergence theorem $\tau^n(0,y) \underset{n\to\infty}{\nearrow} \tau(0,y)$, and thus $\hat{\chi}_n(\beta) \nearrow \chi(\beta)$, or

$$\lim_{n\to\infty}\hat{\chi}_n(\beta)^{-1} = \chi(\beta)^{-1} \tag{3.16}$$

It is easy to see that the functions $\hat{\chi}_n(\beta)$, which refer to finite systems, are piecewise differentiable in β. Using (3.9) we get

$$\begin{aligned} \frac{d}{d\beta}\hat{\chi}_n(\beta) &\leqslant \sup_{x\in\Lambda_n}\frac{d}{d\beta}\sum_{y\in\Lambda_n}\tau^n(x,y) \\ &\leqslant \sup_{x\in\Lambda_n}\sum_{u,v,y\in\Lambda_n}\tau^n(x,u)\left[\frac{d}{d\beta}K_{\{u,v\}}(\beta)\right]\tau^n(v,y) \\ &\leqslant \left[\frac{d}{d\beta}\overline{K}(\beta)\right]\hat{\chi}_n(\beta)^2 \end{aligned} \tag{3.17}$$

An efficient way to write (3.17) is as follows:

$$-\frac{d}{d\beta}\hat{\chi}_n(\beta)^{-1} \leqslant \frac{d}{d\beta}\overline{K}(\beta) \tag{3.18}$$

The uniform bound (3.18), and (3.16) imply both the continuity of $\chi(\beta)^{-1}$ and (3.3) [i.e., (3.18), in the weak form, for the limiting function]. ■

The results and the arguments of Section 5 show that in a large class of percolation models, for each $\beta < \beta_c$, there is a finite correlation length $\xi(\beta) \in (0, \infty)$ with which

$$\tau(x, y) \leqslant e^{-|x-y|/\xi} \tag{3.19}$$

where $|x - y|$ is a T-invariant metric on $\mathbb{L}$. Proposition 3.1 has the following important consequence.

Corollary 3.1. If (3.19) is satisfied, for $\beta < \beta_c$, on an infinite lattice $\mathbb{L}$, with a metric for which $\sum_{x \in \mathbb{L}} e^{-|x|\epsilon} < \infty$ $\forall \epsilon > 0$, then

$$\lim_{\beta \nearrow \beta_c} \xi(\beta) = \infty \tag{3.20}$$

Remark. The arguments introduced in the proof of Lemma 3.1 provide also a simple (the "simplest"?) way to prove the vanishing of the "mass-gap" ($m = \xi^{-1}$) in Ising models, and other ferromagnetic systems—for which the analog of Lemma 3.1 holds by the Lebowitz inequality.

3.2. Discussion of the Upper-Critical Dimension

In the previous discussion we found it useful to consider the quantity $\chi(\beta)^{-1}$. In particular, the Bethe lattice law, $\chi(\beta) \cong c/(\beta_c - \beta)$, can be simply characterized by the nonvanishing of the quantity $(d/d\beta)$ $\chi(\beta)^{-1}|_{\beta_c - 0}$. The formula (3.11) leads to the following exact expression for the derivative of $\chi(\beta)^{-1}$ in homogeneous systems:

$$-\left[\frac{d}{d\beta}\overline{K}(\beta)\right]^{-1}\frac{d}{d\beta}\chi(\beta)^{-1}$$
$$= \frac{\sum_{u,x,y \in \mathbb{L}}\left[dK_{\{0,u\}}/d\beta\right]\text{Prob}(\tilde{C}^{\{0,u\}}(0) \ni x,\ \tilde{C}^{\{0,u\}}(u) \ni y, \text{ and } \tilde{C}^{\{0,u\}}(0) \cap \tilde{C}^{\{0,u\}}(u) = \emptyset)}{\sum_{u,x,y \in \mathbb{L}}\left[dK_{\{0,u\}}/d\beta\right]\text{Prob}(C(0) \ni x)\text{Prob}(C(u) \ni y))} \tag{3.21}$$

The summation over u in (3.21) is effectively restricted to sites near 0, whereas the region over which the x, y sum is significant for the denominator diverges as $\beta \uparrow \beta_c$. Furthermore, as we saw in (3.11), (3.12), each term in the numerator is bounded by the corresponding term in the denominator. A brief contemplation of the ratio of corresponding terms, reveals that the vanishing of $d\chi(\beta)^{-1}/d\beta|_{\beta_c}$ would be closely related with the inability of two large ("incipient") clusters, which reach close sites $(0, u)$, to avoid each other. For comparison, let us mention that the probability that the paths generated by two independent random walks on $\mathbb{Z}^d$ *avoid* each other, vanishes only up to $d = 4$ dimensions. The incipient clusters, whose structure we are in effect probing in the next section, do not look like random walk paths. However, it is not unreasonable to expect that the incipient clusters have a canonical (i.e., d-independent) structure which is unfolded over $\mathbb{L}$, provided the dimension of $\mathbb{L}$ is large enough.

The above picture suggests that for sufficiently high-dimensional $\mathbb{L}$,

$$\left|\frac{d\chi^{-1}}{d\beta}(\beta_c - 0)\right| \neq 0 \tag{3.22}$$

in which case one has a strict equality:

$$\gamma = 1 \tag{3.23}$$

The analysis of the next section leads to the following criterion, which is proven in Section 6.

Proposition 3.2. (3.22) and (3.23) are satisfied (the former in the sense of $\liminf_{\beta \nearrow \beta_c}$) for the standard model of $\mathbb{Z}^d$ in any dimension at which

$$\nabla \stackrel{\text{def}}{=} \sum_{x,y} \tau(0,x)\tau(x,y)\tau(y,0) < \infty \qquad \text{at} \quad \beta = \beta_c \tag{3.24}$$

(or, equivalently, ∇ is *uniformly* bounded for $\beta < \beta_c$).

It is interesting to note that an analogous, yet significantly different result holds for the Ising model, and other ferromagnetic spin systems—for which the criterion for the magnetic susceptibility exponent γ to be 1 is the finiteness of the "bubble diagram":

$$B = \sum_x S(0,x)^2 < \infty \qquad \text{at} \quad \beta = \beta_c \tag{3.25}$$

where S is the pair correlation function.[1,9] (There are also similarities in terms of the geometric picture described above.)

The two criteria were compared, and contrasted, in the introduction, with the help of the Fourier-transform representations (1.12)–(1.14). For these, and other considerations (e.g., in Section 6) it is useful to note that $\hat{\tau}(k)$ is positive, by the following general argument.

Lemma 3.3. The function $\tau(x, y)$ is of *positive type*, in *any* percolation model (i.e., not even a necessarily independent one). In particular, for a translation-invariant model on $\mathbb{Z}^d$,

$$\hat{\tau}(k) \geqslant 0 \tag{3.26}$$

Proof. For any summable function $f:\mathbb{L}\to\mathbb{C}$,

$$\sum_{x,y\in L}\bar{f}(x)\tau(x,y)f(y)=\left\langle\sum_{x,y}\bar{f}(x)I[x\ \&\ y\text{ are connected}]f(y)\right\rangle$$
$$=\left\langle\sum_{C}\left|\sum_{x\in C}f(x)\right|^2\right\rangle\geqslant 0 \tag{3.27}$$

where the first sum on the right-hand side is over the set of (random) clusters and $\langle\ \rangle$ denotes the expectation value.

(3.27) implies the stated positivity. Furthermore, via standard arguments it yields the following intriguing representation for $\hat{\tau}(k)$, at $\beta<\beta_c$:

$$\hat{\tau}(k)=\sum_{A\subset\mathbb{Z}^d}\operatorname{Prob}(C(0)=A)\left||A|^{-1/2}\sum_{x\in A}e^{i(k,x)}\right|^2 \tag{3.28}$$

where $|A|$ denotes the cardinality of A. ■

The analogy with spin systems seems to stop here. The important bound (1.15), whose consequences (and "would be" consequences) were discussed above, has no known analog in percolation models. Furthermore, there are certain indications that in some low dimensions $\hat{\tau}(k)\simeq$ const$\,k^{-(2-\eta)}$ with a *negative* η, i.e., the simple analog of (1.15) fails.[12] (We thank J. Adler, A. B. Harris, and Y. Shapir for bringing this to our attention.) With the above definition of the critical exponent η we can say that, in any dimension, $\eta>(6-d)/3$ implies $\gamma=1$.

4. TREE GRAPH BOUNDS FOR THE CONNECTIVITY FUNCTIONS

4.1. Description of the Main Result

The previous discussion focussed on the behavior of the two-point function $\tau(x,y)$ $(=\tau_2(x,y))$ in the critical regime. However, in addition to that one would like to understand also the structure of the higher connectivity functions, which are defined as follows:

$$\tau_n(x_1,\ldots,x_n)=\operatorname{Prob}(x_1,\ldots,x_n\text{ all belong to the same connected cluster}) \tag{4.1}$$

The functions τ_n contain further (in fact, all the) information about the structure of the connected clusters. In particular, the moments of $|C(0)|$ are given by

$$\langle|C(0)|^n\rangle=\left\langle\left\{\sum_{x\in\mathbb{L}}I[x\in C(0)]\right\}^n\right\rangle$$
$$=\sum_{x_1,\ldots,x_n\in\mathbb{L}}\tau_{n+1}(0,x_1,\ldots,x_n) \tag{4.2}$$

The main results of this section are the "tree diagram bounds," introduced below, for which $\mathbb{L}$ need not be homogeneous (nor infinite). First let us define the following functions.

Definition. For any $n > 2$ and $x_1, \ldots, x_n \in \mathbb{L}$,

$$T_n(x_1, \ldots, x_n) = \sum_G{}^1 \sum_{y_1, \ldots, y_{n-2} \in \mathbb{L}} \prod_{\{z,z'\} \in \mathscr{E}(G)} \tau(z, z') \tag{4.3}$$

where the sum$\sum^1$ is over all the connected tree graphs (i.e., with no loops) whose vertex set is the set of variables $\{x_1, \ldots, x_n, y_1, \ldots, y_{n-2}\}$, such that the number of edges to which a vertex belongs is exactly *one* for the "external" vertices $\{x_1, \ldots, x_n\}$, and *three* for the "internal" vertices $\{y_1, \ldots, y_{n-2}\}$. $\mathscr{E}(G)$ is the set of edges, which are identified in the above expression with the corresponding pairs of (possibly equal) sites of $\mathbb{L}$.

To remove the redundancy which will be associated with coincident points (whose contribution is actually negligible as $\beta \uparrow \beta_c$) let us also denote

$$T_n'(x_1, \ldots, x_n) = \sum_{\substack{V \subset \mathbb{L} \setminus \{x_1, \ldots, x_n\} \\ |V| \leqslant n-2}} \sum_G{}^2 \prod_{\{z,z'\} \in \mathscr{E}(G)} \tau(z, z') \tag{4.4}$$

where the sum $\sum^2$ is over all the connected tree graphs on the vertex set $\{x_1, \ldots, x_n\} \cup V$ such that each of the vertices, $y \in V$, belongs to at least three edges and each of the vertices x_i belongs to at least one edge.

Proposition 4.1. In *any* independent percolation model

$$\tau_3(x_1, x_2, x_3) \leqslant \sum_y \tau(y, x_1)\tau(y, x_2)\tau(y, x_3) \tag{4.5}$$

and, more generally,

$$\tau_n(x_1, \ldots, x_n) \leqslant T_n(x_1, \ldots, x_n) \tag{4.6}$$

[It is clear from the proof of Lemma 4.1 that T_n could be replaced, in (4.6), by T_n' ($\leqslant T_n$).]

Before proving the proposition let us present some heuristic ideas about the structure which emerges here.

The bounds (4.5), (4.6) are made somewhat intuitive by considering first the low-β limit, in which the bond occupation probabilities are very small. In this case, the main contribution to τ_n is from the "minimal" configurations of bonds which interconnect $x_1, \ldots, x_n$. Such configurations correspond to the dominating terms in the sum T_n. For noncoincident points one has $\lim_{\beta \to 0} \tau_n(x_1, \ldots, x_n)/T_n(x_1, \ldots, x_n) = 1$.

For higher values of β, a similar description should still be correct (below β_c), if applied to intermediate size clusters—provided the "interaction" between such "neighboring" clusters is not singular, i.e., the probability of intersection is not 1. We expect that to be the case above the upper critical dimension discussed in the preceding section.

The above considerations, and the proof of (4.6), lead us to the following conjecture, for which we choose as a concrete criterion the behavior of the quantity:

$$Y = \sum_{x_1, x_2 \in \mathbb{L}} \tau_3(0, x_1, x_2) \qquad \left[\leqslant \chi^3, \text{ by } (4.5) \right] \tag{4.7}$$

Conjecture. If for a percolation model on $\mathbb{Z}^d$, with $K_{\{x,y\}}(\cdot)$ of finite range, the ratio Y/χ^3 has nonvanishing limit:

$$G \stackrel{\text{def}}{=} \lim_{\beta \uparrow \beta_c} \frac{Y(\beta)}{\chi(\beta)^3} > 0 \tag{4.8}$$

then, for any (noncoincident) $x_1, \ldots, x_n \in \mathbb{Z}^d$,

$$\lim_{\substack{\beta \uparrow \beta_c \\ \zeta \uparrow \infty}} \frac{\tau_n(x_1 \zeta, \ldots, x_n \zeta)}{T_n(x_1 \zeta, \ldots, x_n \zeta)} = G^{n-2} \tag{4.9}$$

where ζ can be any function of β which diverges when $\beta \uparrow \beta_c$. In (4.9) $x\zeta$ should be interpreted as the closest site in $\mathbb{Z}^d$ to the given point.

We expect that the method of Section 6 can be used to show that in systems in which the high dimensionality criterion (3.24), of Proposition 3.2, is met, there is also a lower bound of the form $\tau_n / T_n \geqslant \delta^{n-2}$ with $\delta > 0$.

When (4.9) holds, the higher connectivity functions reduce to simple combinations of the two-point function—given by tree diagrams with vertices of order 3, and vertex strength G. These diagrams have the appearance of the tree diagrams of a ϕ^3 field theory. A relation between percolation and the ϕ^3 field theory has indeed been expected, on the basis of arguments (see Ref. 15), which we find far less compelling than even the above heuristic discussion.

Other implications of (4.6) are discussed in Section 5.1. Let us now turn to the proof.

4.2. Proof and Some Other Useful Inequalities

In our derivations of various inequalities a key role is played by certain random subsets of $\mathbb{L}$ [an example of which is $C(x_1)$] with a locality property analogous to the nonanticipating property possessed by "stopping times" in the classical theory of random walks, Markov processes, and martingales.

Since various examples of such random sets have been previously employed in the study of percolation and related problems, it is useful to formalize their general structure. We consequently offer the following definitions.

Definition. (a) A set valued function $S(\{n_b\}) \subset \mathbb{L}$ is a *random subset* of $\mathbb{L}$, if for each finite $A \subset \mathbb{L}$ the indicator function $I[S(\{n_b\}) \subset A]$ is a measurable function of $\{n_b\}$. [If this part of the definition causes pain, one need not worry—we shall not see here nonmeasurable functions of $\{n_b\}$.]

(b) S, a random subset of $\mathbb{L}$, is said to be *self-determined*, if for each nonrandom $A \subset \mathbb{L}$ (possibly infinite), the event $\{S \subset A\}$ is determined by the values of $\{n_b\}$ for those bonds which have at least one end point in A (i.e., $\{S \subset A\}$ is in the σ-field generated by the above described set of bond variables).

In the last statement we could also refer directly to the events $\{S = A\}$, however for infinite sets A such statements require a proper interpretation.

Our use of self-determined sets is based on the fact that for independent percolation the conditional distribution of the occupation variables, conditioned on $\{S = A\}$, remains unchanged for the bonds which are external to A. An application is seen in the result presented next, which, in addition to being used in the proof of Proposition 4.1, is also of independent interest (see Section 5).

Definition. We say that a set $V \subset \mathbb{L}$ is *connected in* Λ, a subset of $\mathbb{L}$, if V is connected by the set of those occupied bonds whose end points lie (both) in Λ. The probability of such an event is denoted by $\tau_\Lambda(V)$. In particular, $\tau_\Lambda(V) = 0$ unless $V \subset \Lambda$, and

$$\tau(V) \stackrel{\text{def}}{=} \tau_{\mathbb{L}}(V) = \tau_n(x_1, \ldots, x_n)$$

for $V = \{x_1, \ldots, x_n\}$.

Proposition 4.2. Let $A \subset \bar{A} \subset \mathbb{L}$. Then for every $V = \{x_1, \ldots, x_n\} \subset \mathbb{L}$

$$0 \leqslant \tau_{\mathbb{L}\setminus A}(V) - \tau_{\mathbb{L}\setminus \bar{A}}(V) \leqslant \sum_{y \in \bar{A}\setminus A} \sum_{\substack{x_1 \in W \subset V \\ |W| < |V|}} \tau(W \cup \{y\})\tau((V\setminus W) \cup \{y\}) \tag{4.10}$$

Proof. It suffices to prove (4.10) for $|\bar{A}\setminus A| = 1$, and $|\mathbb{L}| < \infty$. The general case follows by a simple telescopic decomposition of $\tau_{\mathbb{L}\setminus A} - \tau_{\mathbb{L}\setminus \bar{A}}$ associated with an interpolating sequence $A = A_0 \subset A_1 \subset A_2 \subset \cdots \subset \bar{A}$, with $|A_{i+1}\setminus A_i| = 1$. The result extends to infinite $\mathbb{L}$ by a simple continuity argument. However, it should be pointed out that with a proper introduction, the following argument applies also directly to infinite systems.

Thus, we assume that $\bar{A} = A \cup \{y\}$. Let S be the random set of sites which are connected to x_1 in $\mathbb{L}\setminus\bar{A}$. It is easy to see that

$$\begin{aligned}
&\tau_{\mathbb{L}\setminus A}(V) - \tau_{\mathbb{L}\setminus\bar{A}}(V)\\
&\quad = \langle I[V \text{ is connected in } \mathbb{L}\setminus A]\rangle - \langle I[V \text{ is connected in } \mathbb{L}\setminus\bar{A}]\rangle\\
&\quad = \langle I[V \text{ is connected in } \mathbb{L}\setminus A \text{ but not in } \mathbb{L}\setminus\bar{A}]\rangle\\
&\quad = \sum_{x_1 \in W \subsetneqq V} \langle I[S \cap V = W]\\
&\qquad\qquad \times I[\text{at least one of the bonds between } y \text{ and } S \text{ is occupied}]\\
&\qquad\qquad \times I[(V\setminus W) \cup \{y\} \text{ is connected in } \mathbb{L}\setminus(A \cup S)]\rangle
\end{aligned} \tag{4.11}$$

The random set S is clearly self-determined. Conditioning on those bonds which "touch" S, we obtain

$$\begin{aligned}
0 &\leqslant \tau_{\mathbb{L}\setminus A}(V) - \tau_{\mathbb{L}\setminus\bar{A}}(V)\\
&= \sum_{x_1 \in W \subsetneqq V} \langle I[S \cap V = W] I[y \text{ is directly connected to } S]\\
&\qquad\qquad \times \tau_{\mathbb{L}\setminus(A \cup S)}((V\setminus W) \cup \{y\})\rangle\\
&\leqslant \sum_{x_1 \in W \subsetneqq V} \tau(W \cup \{y\})\tau((V\setminus W) \cup \{y\})
\end{aligned} \tag{4.12}$$

As explained above, this proves (4.10). ∎

As a final prelude to Proposition 4.1, let us present the following somewhat stronger statement.

Lemma 4.1. In any independent bond percolation model

$$\begin{aligned}
\tau(\{x_1, \ldots, x_n\}) \leqslant \sum_y \sum_{x_2 \in W \subsetneqq \{x_2, \ldots, x_n\}} &\tau(x_1, y)\tau(W \cup \{y\})\\
&\times \tau((\{x_2, \ldots, x_n\}\setminus W) \cup \{y\})
\end{aligned} \tag{4.13}$$

Proof. By simple "logic,"

$$\begin{aligned}
&\tau(\{x_1, \ldots, x_n\})\\
&\quad = \langle I[\{x_1, x_2, \ldots, x_n\} \text{ is connected}]\rangle\\
&\quad = \langle I[\{x_2, \ldots, x_n\} \text{ is connected}]\\
&\qquad - I[\{x_2, \ldots, x_n\} \text{ is connected in } \mathbb{L}\setminus C(x_1)]\rangle\\
&\quad = \langle \tau_{\mathbb{L}}(\{x_2, \ldots, x_n\}) - \tau_{\mathbb{L}\setminus C(x_1)}(\{x_2, \ldots, x_n\})\rangle
\end{aligned} \tag{4.14}$$

Substituting in (4.14) the bound (4.10), we obtain

$$\tau(\{x_1, \ldots, x_n\}) \leqslant \sum_{y \in \mathbb{L}} \sum_{x_2 \in W \subsetneqq \{x_2, \ldots, x_n\}} \langle I[y \in C(x_1)] \tau(W \cup \{y\}) \times \tau((\{x_2, \ldots, x_n\} \backslash W) \cup \{y\}) \rangle \tag{4.15}$$

which leads directly to (4.13). ■

Remark. A direct proof, and a simple grasp, of Lemma 4.1 may be obtained by considering the following sequential, self-determined, decomposition of $C(x)$ which is modeled after algorithmic constructions of percolation clusters used in Ref. 16.

For the construction, let us first choose an arbitrary total ordering of $\mathbb{L}$ (e.g., a "spiral" ordering for $\mathbb{L} = \mathbb{Z}^d$). Using it, we define for each $x \in \mathbb{L}$, and a given configuration of occupied bonds:

$$C_1(x) = \{x\}$$

$$C_n(x) = C_{n-1}(x) \cup \{\text{the "earliest" site in } \mathbb{L} \backslash C_{n-1}(x) \text{ which shares an occupied bond with some site in } C_{n-1}(x)\}$$

for $1 < n \leqslant |C(x)|$, and

$$C_n(x) = C_\infty(x) = C(x) \qquad \text{for} \quad n > |C(x)|$$

A moment's reflection shows that (i) $C_n(x)$ is self-determined for each (nonrandom) $n \in \mathbb{Z}_+ \cup \{\infty\}$, $x \in \mathbb{L}$, and (ii) $C_n(x) \nearrow C(x)$ (local convergence) as $n \nearrow \infty$.

A direct proof of Lemma 4.1 is obtained by noting that if $\{x_1, \ldots, x_n\}$ are connected then there is some $1 \leqslant k < \infty$ at which $\{x_2, \ldots, x_n\}$ ceases being connected in $\mathbb{L} \backslash C_k(x)$. The terms in (4.13) with a given $y \in \mathbb{L}$ are a bound on the probability that $C_k(x_1) \backslash C_{k-1}(x_1) = y$. The bound is derived as in (4.12), with $S = C_{k-1}(x_1)$.

Finally, let us finish the proof of the result discussed in Section 4.1.

Proof of Proposition 4.1. Lemma 4.1 provides a bound on τ_n, for $n > 2$, in terms of strictly lower-order connectivity functions. Repeated substitution of (4.13) in its right-hand side leads (in $n-3$ steps) to an expression which involves only τ_2. The resulting sum is easily seen to be the tree diagram bound claimed in (4.6). ■

It is interesting to note that the deviations from equality in (4.5) and (4.6) are fully traceable to the replacement of quantities like $\tau_{\mathbb{L} \backslash S}(\cdot)$, or $\tau_{\mathbb{L} \backslash C_{k-1}}(\cdot)$ in the above direct proof, by $\tau(\cdot)$. Implications of the observation were mentioned in the discussion at the beginning of this section.

5. EXPONENTIAL DECAY AND RELATED INEQUALITIES

Two types of exponential decay are considered in this section. The first is a general result on the *cluster size* distribution for $\beta < \beta_c$, valid even for long range percolation (e.g., of the type considered in Ref. 8). In particular, this result simplifies and improves a previous result of Kesten on finite-range percolation.[13] The other, presented in Section 5.3, concerns improved bounds on the exponential decay of the connectivity functions τ_n for short-range systems. Some inequalities used for the latter are presented in Section 5.2.

5.1. Exponential Decay in the Cluster Size Distribution

The tree diagram bounds have the following implication.

Proposition 5.1. Suppose, in an independent percolation model

$$\chi = \sup_{y \in \mathbb{L}} \langle |C(y)| \rangle < \infty \tag{5.1}$$

then, for every $x \in \mathbb{L}$ and $k \geqslant \chi^2$,

$$\operatorname{Prob}(|C(x)| \geqslant k) \leqslant (e/k)^{1/2} e^{-k/(2\chi^2)} \tag{5.2}$$

Proof. The moment formula (4.2), and the bound (4.6) imply that

$$\langle |C(x)|^n \rangle = \sum_{y_1, \ldots, y_n} \tau_{n+1}(x, y_1, \ldots, y_n) \leqslant N_{n+1} \chi^{2n-1} \tag{5.3}$$

where N_{n+1} is the number of tree graphs appearing in T_{n+1}. It is easy to see that N_{n+1}/N_n is the number of edges in the tree graphs of T_n, which is $(2n-3)$. Therefore

$$N_{n+1} = (2n-3)!! = (2n-2)!/\left[2^{n-1}(n-1)!\right] \tag{5.4}$$

Summing (5.3) with weights given by the corresponding power expansion, we get (with no further loss)

$$\langle |C| e^{r|C|} \rangle \leqslant \chi(1 - 2\chi^2 r)^{-1/2} \tag{5.5}$$

for $r < (2\chi^2)^{-1}$, $|C| \equiv |C(x)|$.

By a variant of Tchebyshev's inequality, (5.5) implies that

$$\begin{aligned} \operatorname{Prob}(|C(x)| \geqslant k) &\leqslant \inf_{r \geqslant 0} \langle |C| e^{r|C|} \rangle / (k e^{rk}) \\ &\leqslant \frac{\chi}{k} \inf_{r \geqslant 0} (1 - 2\chi^2 r)^{-1/2} e^{-rk} \end{aligned} \tag{5.6}$$

which [with $r = (2\chi^2)^{-1} - (2k)^{-1}$] yields (5.2). ∎

To complement the bound (5.2) let us mention that the behavior of $\mathrm{Prob}(\infty > |C(0)| \geqslant k)$ is qualitatively different for high β.[17,18]

In Section 4.1 a conjecture was made about the structure of the connectivity functions "above the upper critical dimension." For the moments of $|C(0)|$, the behavior which corresponds to (4.9) is

$$\lim_{\beta \uparrow \beta_c} \frac{\langle |C(0)|^n \rangle}{\chi^{2n-1}} = N_{n+1} G^{n-1} \tag{5.7}$$

It is interesting to note that if one defines a random variable W with

$$\mathrm{Prob}(W = k) = k\, \mathrm{Prob}(|C(0)| = k)/\chi \tag{5.8}$$

then (5.7) corresponds to the statement that

$$\lim_{\beta \uparrow \beta_c} W/\chi^2 = GZ^2 \tag{5.9}$$

where Z is a standard normal random variable and the limit is in the sense of convergence in distribution.

It can be shown that (5.7) is satisfied in any percolation model on a "rootless" Bethe lattice, i.e., $\mathbb{L} = \{(j,m) \,|\, j \in Z_+,\ m = 1, \ldots, K^j\}$ with $K > 1$ and the usual bond structure. (This fact was also noted by Durrett[4].)

For systems with a finite-range function $K_b(\beta)$, (5.2) can be used to easily obtain exponential decay of $\tau(x, y)$ in the distance $\|x - y\|$. However a better decay constant will be obtained in Section 5.3, by using the analog of the Simon–Lieb inequality which is derived next.

5.2. Inequalities of Simon–Lieb Type

We now turn our attention back to the two-point function $\tau(x, y)$, starting with the derivation of a number of inequalities analogous to the Simon–Lieb correlation inequalities for Ising models. Following the discovery of the existence and usefulness of the original inequalities for ferromagnetic spin systems,[6,7] the existence of such inequalities for percolation models was realized by a number of people—including B. Souillard and F. Delyon,[19] A. Sokal,[20] and J. Fröhlich.[2]

The first inequality is really a special case of Proposition 4.2.

Corollary 5.1. In an independent bond percolation model

$$\tau(x,z) - \tau_{\mathbb{L}\setminus A}(\{x,z\}) \leqslant \sum_{y \in A} \tau(x, y)\tau(y, z) \tag{5.10}$$

for every $x, z \in \mathbb{L}$, $A \subset \mathbb{L}$.

Note that if the set A *separates* the two points x, z—in the sense that any connecting path along bonds with $K_b \neq 0$ intersects A—then

$$\tau_{\mathbb{L}\setminus A}(\{x,z\}) = 0 \tag{5.11}$$

The restriction of (5.10) to such separating sets yields a direct analog of the Simon inequality (of Ref. 6).

For an improvement, somewhat analogous to Lieb's,[7] let us define

$$\begin{aligned}\hat{\tau}^A(x,y) = \text{Prob}(&x \text{ and } y \text{ are connected by a path of}\\ &\text{occupied bond, of which not more}\\ &\text{than one touches } A)\end{aligned} \tag{5.12}$$

Proposition 5.2. For each $x, z \in \mathbb{L}$ and $A \subset \mathbb{L}$,

$$\tau(x,z) - \tau_{\mathbb{L}\setminus A}(\{x,z\}) \leqslant \sum_{y\in A} \hat{\tau}^A(x,y)\tau(y,z) \tag{5.13}$$

Proof. The proof follows the approach used in Proposition 4.2. For reasons mentioned there, it suffices to deal with finite systems.

Let S be the random set

$$S = \{v \in \mathbb{L} \mid v \text{ is connected to } x \text{ in } \mathbb{L}\setminus A\} \tag{5.14}$$

Then, as in (4.11),

$$\begin{aligned}
&\tau(x,z) - \tau_{\mathbb{L}\setminus A}(\{x,z\})\\
&\quad = \langle I[x \text{ and } z \text{ are connected in } \mathbb{L} \text{ but not in } \mathbb{L}\setminus A]\rangle\\
&\quad = \langle I[S \not\ni z] I[\text{there is some point } y \in A \text{ which is connected}\\
&\qquad\qquad \text{to } z \text{ in } \mathbb{L}\setminus S \text{ and which is linked to } S\\
&\qquad\qquad \text{by an occupied bond}]\rangle\\
&\quad \leqslant \sum_{y\in A} \langle I[y \text{ is directly connected to } S]\tau(y,z)\rangle\\
&\quad = \sum_{y\in A} \hat{\tau}^A(x,y)\tau(y,z). \quad \blacksquare
\end{aligned} \tag{5.15}$$

Remark. To relate $\hat{\tau}^A$ to the usual connectivity function, let

$$\begin{aligned}\mathbb{L}(x,A) = \{u \in \mathbb{L} \mid &\text{ there is a path from } u \text{ to } x \text{ which has}\\ &\text{not more than one bond touching } A\}\end{aligned}$$

It is easy to see that

$$\hat{\tau}^A(x,y) \leqslant \tau_{\mathbb{L}(x,A)}(\{x,y\}) \tag{5.16}$$

In certain situations (5.16) is not a very wasteful inequality. However, that is certainly not the case if the set A does not "enclose" any volume.

Finally, let us mention that related bounds exist for the function $\tilde{\tau}^B(x, y)$, defined for B—a set of *bonds*, as the probability that x and y are connected even *without* the bonds in B.

One can show, by the arguments used above, that

$$\tau(x,z) - \tilde{\tau}^B(x,z) \leqslant \sum_{\substack{u,v \in \mathbb{L} \\ \{u,v\} \in B}} \tilde{\tau}^B(x,u) K_{\{u,v\}}(\beta) \tau(v,z) \tag{5.17}$$

(5.17) may be used for another proof of Lemma 3.2. We shall next see another application of such inequalities.

5.3. Exponential Decay of $\tau(0,x)$

For finite-range percolation models, the above inequality can be used to prove the exponential decay of $\tau(0,x)$, for $\beta < \beta_c$, with an explicit estimate of the exponential decay rate (i.e., the "connectivity length").

In a general percolation model, let $\mathscr{B}$ be the set of "relevant" bonds: $\mathscr{B} = \{b \mid K_b(\cdot) \not\equiv 0\}$, and let ρ denote the following metric:

$\rho(x, y) =$ the minimal number of bonds in $\mathscr{B}$ needed to connect x with y

In a homogeneous model we denote, for $x \in \mathbb{L}$:

$$\|x\| = \sup\left\{ \lim_{k\to\infty} \frac{\rho(0, T_x^k 0)}{k} \,\middle|\, T_x \text{ is a lattice translation such that } T_x 0 = x \right\} \tag{5.18}$$

[The limit(s) exist by subadditivity.]

For example, in the standard nearest-neighbor model on $\mathbb{Z}^d$

$$\|x\| = \sum_{i=1}^{d} |x_i|, \qquad \text{for all} \quad x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \tag{5.19}$$

The general result (which may be further extended to nonhomogenous cases) is

Proposition 5.3. In any homogeneous independent bond percolation model, with $\chi < \infty$,

$$\tau(0,x) \leqslant (1 - \chi^{-1})^{\|x\|} \leqslant e^{-\|x\|/\chi} \tag{5.20}$$

Proof. Let

$$g_n = \sum_{\substack{y \in \mathbb{L} \\ \rho(0,y) = n}} \tau(0, y)$$

By Corollary 5.1 with $A = \{y \mid \rho(y,x) = n\}$, and (5.11) with $x = 0$, $z = x$, we have

$$\tau(0,x) \leqslant \sum_{\substack{y \in \mathbb{L} \\ \rho(y,x) = n}} \tau(0,y)\tau(y,x) \tag{5.21}$$

for any x such that $\rho(0,x) \geqslant n$. A simple iteration of this inequality (as in Ref. 6) shows that

$$\tau(0,x) \leqslant g_n^{[\rho(0,x)/n]}, \qquad \text{if} \quad \rho(0,x) \geqslant n \tag{5.22}$$

where $[a] \geqslant a - 1$ is the integral part of a.

Lemma 5.1, below, implies that (5.22) can in fact be simplified into

$$\tau(0,x) \leqslant \Big(\inf_{n \geqslant 1} g_n^{1/n} \Big)^{\|x\|}, \qquad \text{for all} \quad x \in \mathbb{Z}^d \tag{5.23}$$

The necessary bound on inf $g_n^{1/n}$ is provided by Lemma 5.2. ■

We referred above to the following results.

Lemma 5.1. For any homogeneous independent bond percolation model

$$\tau(0,x) \leqslant \inf_{k \geqslant 0} \tau\big(0, T_x^k 0\big)^{1/k} \tag{5.24}$$

for any translation T_x such that $T_x 0 = x$ (in the standard models: $T_x^k 0 = kx$).

Proof. Given a translation T_x, with $T_x 0 = x$, it is natural to denote $kx = T_x^k 0$. For each integer $k \geqslant 0$ we have

$$\begin{aligned} \tau(0,kx) &\geqslant \tau(\{0,x,2x,\ldots,kx\}) \\ &\geqslant \tau(0,x)\tau(x,2x)\ldots\tau((k-1)x,kx) = \tau(0,x)^k \end{aligned} \tag{5.25}$$

where the second step is by the FKG inequality. ■

Lemma 5.2. For any sequence g_n with $g_0 = 1$, $g_n \geqslant 0$ and $\sum_{n=0}^{\infty} g_n = \chi < \infty$, we have

$$\inf_{n \geqslant 1} (g_n)^{1/n} \leqslant 1 - \chi^{-1} \tag{5.26}$$

Proof. If $\inf_{n \geqslant 1}(g_n)^{1/n} > 1 - \chi^{-1}$, then $g_n > (1 - \chi^{-1})^n$ for each $n \geqslant 1$ and thus

$$\sum_{n=0}^{\infty} g_n > 1 + \sum_{n=1}^{\infty} (1 - \chi^{-1})^n = \chi \tag{5.27}$$

which contradicts the given data. ■

Let us conclude this section with a few remarks about percolation models on $\mathbb{Z}^d$.

(1) For a nondegenerate finite range model the norm $\|x\|$ used above is equivalent to the Euclidean norm $|x|$, and satisfies $\|x\| \geqslant c|x|$ for some $c > 0$. Typically, the connectivity (or correlation) length ξ is defined as the *minimal* value for which

$$\tau(0, x) \leqslant e^{-|x|/\xi} \tag{5.28}$$

(by Lemma 5.1, the minimum is attained). Proposition 5.3 shows that for $\beta \uparrow \beta_c$

$$\xi / R \leqslant \left[-\ln(1 - \chi^{-1}) \right]^{-1} \leqslant \chi \tag{5.29}$$

where $R = \max\{|y| \mid \{0, y\} \in \mathscr{B}\}$.

(2) By Simon's argument,[6] the Lieb[7] type improvement made in Proposition 5.2 leads to a proof that

$$\lim_{\beta \uparrow \beta_c} \xi(\beta) = \infty \tag{5.30}$$

However, in Section 3 we presented an even simpler proof of (5.30) [there (3.20)].

(3) For long-range percolation models (as the one studied in Ref. 8), in which $\xi(\beta) = \infty$ for all $\beta > 0$, one may "control" $\tau(x, y)$ by combining the above arguments with a method used in Ref. 21—just as is done in Ref. 6.

6. DERIVATION OF THE ∇-CRITERION

In this section we derive the criterion for the upper critical dimension which was extensively discussed in the introduction and Section 3.2. Without repeating the discussion we shall prove here Proposition 3.2, which may be rephrased as follows.

Proposition 6.1. If, in the nearest-neighbor bond percolation model on $\mathbb{Z}^d$ the triangle diagram is finite at p_c, i.e.,

$$\nabla(p) \stackrel{\text{def}}{=} \sum_{x, y \in \mathbb{L}} \tau(0, x) \tau(x, y) \tau(y, 0) < \infty \qquad \text{at} \quad p = p_c \tag{6.1}$$

(or, equivalently, ∇ is *uniformly* bounded for $p < p_c$) then, for some $\delta > 0$

$$\left| \frac{d\chi(p)^{-1}}{dp} \right| \geqslant \delta \qquad \text{for all} \quad p < p_c \tag{6.2}$$

In order to simplify the presentation, let us prove separately a much less useful result, which shows that (6.2) holds if $\nabla(\beta_c)$ is not just finite but is less than 1.

Lemma 6.1. In any homogeneous bond percolation model

$$\left|\frac{d\chi(\beta)^{-1}}{d\beta}\right| \geqslant \frac{d\bar{K}(\beta)}{d\beta}\left[1-\nabla(\beta)\right] \tag{6.3}$$

Proof. The summation, as in (3.13), of Russo's formula (3.10) leads to the following expression:

$$\begin{aligned}\frac{d\chi(\beta)}{d\beta} &= \sum_{z\in\mathbb{L}} \frac{d}{d\beta}\tau(0,z)\\ &= \frac{1}{2}\sum_{x,y,u\in\mathbb{L}}\left[\frac{d}{d\beta}K_{\{0,u\}}(\beta)\right]\\ &\quad\times\langle I[\{0,u\} \text{ is pivotal for the connection of } x \text{ with } y]\rangle\end{aligned} \tag{6.4}$$

where we made a simple use of the translation invariance (in effect, to simplify later notation we replaced 0 with x in the more natural expression).

Denoting, as in Section 3, by $\tilde{C}(z) \equiv \tilde{C}^{\{0,u\}}(z)$ the cluster of sites connected to z even after the bond $\{0,u\}$ is removed, we have

$$\begin{aligned}&\langle I[\{0,u\} \text{ is pivotal} \ldots]\rangle\\ &\quad= \langle I[C(x)\ni 0]\tau_{\mathbb{L}\setminus\tilde{C}(x)}(u,y)\rangle\\ &\qquad+ a(x\leftrightarrow y) \text{ permutation of the above}\end{aligned} \tag{6.5}$$

Applying Corollary 5.1 (or Proposition 4.2)

$$\begin{aligned}\tau_{\mathbb{L}\setminus\tilde{C}(x)}(u,y) &= \tau(u,y) - \left[\tau(u,y) - \tau_{\mathbb{L}\setminus\tilde{C}(x)}(u,y)\right]\\ &\geqslant \tau(u,y) - \sum_{z\in\mathbb{L}} I[z\in\tilde{C}(x)]\tau(u,z)\tau(z,y)\end{aligned} \tag{6.6}$$

Substituting (6.6) [where $\tilde{C}(x) \subset C(x)$] in (6.5) one gets

$$\begin{aligned}\frac{d\chi(\beta)}{d\beta} \geqslant \sum\left[\frac{d}{d\beta}K_{\{0,u\}}(\beta)\right]&\Big\{\langle I[C(x)\ni 0]\rangle\tau(u,y)\\ &-\sum\langle I[C(x)\ni 0,z]\rangle\tau(u,z)\tau(z,y)\Big\}\end{aligned} \tag{6.7}$$

By definition, and the tree diagram bound (4.5)

$$\langle I[C(x)\ni 0,z]\rangle = \tau_3(0,x,z) \leqslant \sum_w \tau(0,w)\tau(x,w)\tau(z,w) \tag{6.8}$$

(and, of course $\langle I[C(x)\ni 0]\rangle = \tau(0,x)$).

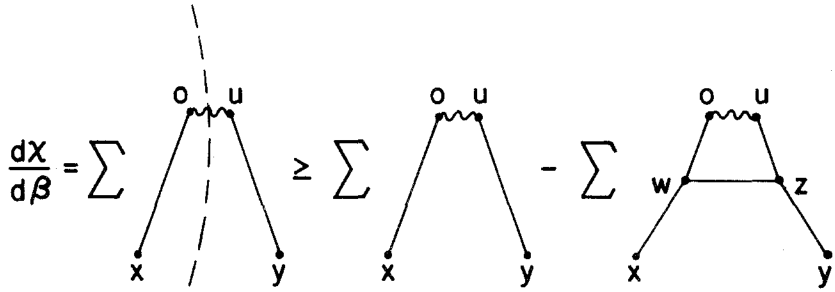

Fig. 2. The lower bound which is obtained by the substitution of (6.8) in (6.7). Each solid line represents a two-point function $\tau(\cdot,\cdot)$, a wiggly line stands for $dK_{\{0,u\}}(\beta)/d\beta$, and the interrupted line indicates that u and v are not connected—except possibly by the bond $\{0,u\}$ itself. The summation is over all the vertices, except 0. (The formula may be easier to recognize after shifting 0 to x.)

The substitution of (6.8) in (6.7) leads to the lower bound which is described in Fig. 2. Its summation is in fact quite simple, due to the translation invariance. The result is

$$\frac{d\chi(\beta)}{d\beta} \geqslant \left[\frac{d\bar{K}(\beta)}{d\beta}\right]\chi^2\left[1 - \sup_{u\in\mathbb{L}} \sum_{w,z} \tau(0,w)\tau(w,z)\tau(z,u)\right] \tag{6.9}$$

The supremum in (6.9) is attained at $u = 0$, by Lemma 6.2. Dividing (6.9) by χ^2 one gets (6.3). ■

In the last step we used the following result.

Lemma 6.2. In a homogeneous model, for every $v \in \mathbb{L}$

$$\sum_{w,z\in\mathbb{L}} \tau(0,w)\tau(w,z)\tau(z,v) \leqslant \sum_{w,z} \tau(0,w)\tau(w,z)\tau(z,0) = \nabla \tag{6.10}$$

Proof. It follows from (3.27), of Lemma 3.3, that the quadratic form with the kernel

$$Q(u,v) = \sum_{w,z\in\mathbb{L}} \tau(u,w)\tau(w,z)\tau(z,v) \tag{6.11}$$

is of positive type. A standard argument, based on the Schwarz inequality, implies that

$$Q(u,v) \leqslant \left[Q(u,u)Q(v,v)\right]^{1/2} = Q(0,0) \tag{6.12}$$

where the last step is by the homogeneity. [For $\mathbb{Z}^d$ one could, alternatively, use the Fourier transform representation, and (3.26).] ■

In order to extend the main result of (6.3) to cases where $\nabla(\beta_c)$ is finite but larger than 1, we shall use the following lemma.

Lemma 6.3. In the nearest-neighbor model on $\mathbb{L} = \mathbb{Z}^d$ $(d > 1)$, for each finite region $\Lambda \supset \{0, u\}$:

$$\begin{aligned}&\langle I[C(x) \ni 0 \text{ and } u \text{ is connected to } y \text{ in } \mathbb{L}\backslash \tilde{C}^{\{0,u\}}(x)]\rangle \\ &\quad \geqslant \epsilon_\Lambda \langle I[C(x) \ni 0] \tau_{\mathbb{L}\backslash \hat{C}^\Lambda(x)}(u, y)\rangle \end{aligned} \tag{6.13}$$

with $\epsilon_\Lambda = [\min\{p, (1-p)\}]^{d|\Lambda|} > 0$, where $\hat{C}^\Lambda(x)$ is the cluster of sites connected to x in $\mathbb{L}\backslash\Lambda$.

Proof. Let the events E, F, and G be defined as follows:

E: $C(x) \ni 0$ and u is connected to y in $\mathbb{L}\backslash \tilde{C}^{\{0,u\}}(x)$ (in which case the bond $\{0, u\}$ is pivotal for the connection of x with y).

F: $C(x) \ni 0$ and u is connected to y in $\mathbb{L}\backslash \hat{C}^\Lambda(x)$.

G: $C(x) \cap \Lambda \neq \emptyset$, $C(y) \cap \Lambda \neq \emptyset$ and $\hat{C}^\Lambda(x) \cap \hat{C}^\Lambda(y) = \emptyset$.

Clearly $G \supset E, F$. Thus

$$\text{Prob}(G) \geqslant \text{Prob}(F) \quad \text{and} \quad \text{Prob}(E) = \text{Prob}(G)\text{Prob}(E \mid G) \tag{6.14}$$

where the last factor is a conditional probability.

The event G depends on only those bonds which do not lie entirely in Λ. It is easy to see that (in $d > 1$ dimensions) for each configuration of bonds in this set which occurs in G, there is some configuration of the complementary set of the nearest-neighbor bonds of Λ with which the event E occurs. Therefore

$$\text{Prob}(E \mid G) \geqslant [\min\{p, (1-p)\}]^{d|\Lambda|} = \epsilon_\Lambda \tag{6.15}$$

and hence, by (6.14),

$$\text{Prob}(E) \geqslant \epsilon_\Lambda \text{Prob}(F) \tag{6.16}$$

which is equivalent to (6.13). ■

Proposition 6.1 will now be proven by the argument of Lemma 6.1, combined with (6.13).

Proof of Proposition 6.1. It clearly suffices to prove (6.2) for the range $p \in (p_c/2, p_c)$. We know of course that $p_c < 1$—however, more generally, that is also a necessary condition for (6.1).

By (6.4), (6.5), and (6.13),

$$\frac{d\chi(p)}{dp} \geqslant \epsilon_\Lambda \sum_{\substack{u,x,y \in \mathbb{Z}^d = \mathbb{L} \\ |u| = 1}} \langle I[C(x) \ni 0] \tau_{\mathbb{L}\backslash \hat{C}^\Lambda(x)}(u, y)\rangle \tag{6.17}$$

In applying the bound (6.6) to (6.17) we may now restrict the summation over z to $\mathbb{L}\backslash\Lambda$. Using (6.8), and summing, as in the proof of Lemma 6.1, one gets

$$\frac{d\chi(p)}{dp} \geqslant \epsilon_\Lambda 2d\chi^2\left[1 - \sup_{|u|=1} \sum_{\substack{w,z \\ z\in\mathbb{L}\backslash\Lambda}} \tau(0,w)\tau(w,z)\tau(z,u)\right] \tag{6.18}$$

with $\epsilon_\Lambda(p) \geqslant [\min\{p_c/2,(1-p_c)\}]^{d|\Lambda|}$—uniformly in $p\in[p_c/2,p_c)$. [$2d$ plays in (6.18) the role of $d\overline{K}(\beta)/d\beta$.]

Since $\nabla(p_c) < \infty$, there is some finite Λ for which

$$\sum_{\substack{w,z \\ z\in\mathbb{L}\backslash\Lambda}} \tau(0,w)\tau(w,z)\tau(z,u) \leqslant 1/2 \tag{6.19}$$

for each u with $|u| = 1$. With this choice of Λ one gets (6.2) (after dividing (6.18) by χ^2) with $\delta = 2d\epsilon_\Lambda/2$. ■

It is clear from the argument that Lemma 6.3 and thus Proposition 6.1 can be extended to more general systems.

7. INEQUALITIES FOR SITE PERCOLATION

7.1. Notation

In this section we present the analogs for site percolation of the results derived and discussed in the previous sections for bond percolation models. Since the proofs are essentially the same, we only sketch them with emphasis on the changes. Those are primarily in the definitions of the relevant self-determined sets.

In an independent site percolation model on a lattice $\mathbb{L}$ the sites in $\mathbb{L}$ are independently occupied, with probability p, or vacant, with probability $1-p$. For each site $x\in\mathbb{L}$ there is an *a priori* specified collection, $N(x)\subset\mathbb{L}$, of "neighbors." In standard models $\mathbb{L}=\mathbb{Z}^d$, and $x, y\in\mathbb{Z}^d$ are neighbors if $|x-y| = 1$. The discussion is confined here to models in which the relation $y\in N(x)$ ("x is a neighbor of y") is symmetric. However, this property is not an essential requirement for our methods, which can also be adapted for "oriented percolation."

It will also be assumed that the following quantity is finite

$$N \stackrel{\text{def}}{=} \sup_{x\in\mathbb{L}} \operatorname{card}(N(x)) < \infty \tag{7.1}$$

We say that x and y are connected (resp. connected in $A\subset\mathbb{L}$), for a specified configuration of the occupation variables, if there is a sequence of neighboring sites $z_1 = x,\ z_2, \ldots, z_k = y (z_{i+1}\in N(z_i))$ all of which are occupied (and, respectively, in A). The set $\{x_1, \ldots, x_n\}$ is said to be connected (or connected in A) if each pair x_i, x_j is connected (resp. connected in A).

We define $C(x)$, the cluster of x, and $\mathscr{L}(x)$, the "augmented" cluster of x, by

$$\begin{aligned} C(x) &= \{ y \in \mathbb{L} \mid x \text{ and } y \text{ are connected} \} \\ \mathscr{L}(x) &= \{x\} \cup \left(\bigcup_{z \in N(x)} C(z) \right) \end{aligned} \tag{7.2}$$

Thus $\mathscr{L}(x)$ is the connected cluster of x in the configuration obtained by setting x as occupied. The connectivity functions associated with these notions are

$$\tau(x_1, \ldots, x_n) = \text{Prob}(\{x_1, \ldots, x_n\} \text{ is connected}) = \tau(\{x_1, \ldots, x_n\}) \tag{7.3}$$

$$\sigma(x_1, \ldots, x_n) = \text{Prob}(x_2, \ldots, x_n \in \mathscr{L}(x_1)) = p^{-1}\tau(x_1, \ldots, x_n)$$

Notice that σ, as well as τ, are symmetric in their arguments. $C_A(x)$, $\tau_A(\{x_1, \ldots, x_n\})$, etc. are defined similarly by the connections in A.

The expected sizes for the clusters, and augmented clusters, are

$$\begin{aligned} \langle |C(x)| \rangle &= \sum_{y \in \mathbb{L}} \tau(x, y) \\ \langle |\mathscr{L}(x)| \rangle &= \sum_{y \in \mathbb{L}} \sigma(x, y) = p^{-1} \langle |C(x)| \rangle \end{aligned} \tag{7.4}$$

We denote

$$\begin{aligned} \chi &= \sup_x \langle |C(x)| \rangle \\ \bar{\chi} &= \sup_x \langle |\mathscr{L}(x)| \rangle = p^{-1}\chi \end{aligned} \tag{7.5}$$

and define the critical density by

$$p_c = \sup\{ p \in [0, 1] \mid \chi(p) < \infty \} \tag{7.6}$$

$\mathscr{L}(x)$ and $\bar{\chi}$ were introduced here since, as will be seen, they offer closer analogies than $C(x)$ and χ to the bond percolation clusters.

A *self-determined set* for site percolation is a random subset S of $\mathbb{L}$ such that for each nonrandom $A \subset \mathbb{L}$, the event $\{S \subset A\}$ (and hence also $S = A$) is determined entirely by the occupation/vacancy of sites in A. Note that $C(x)$ is not a self-determined set, but is closure

$$\bar{C}(x) \stackrel{\text{def}}{=} \{x\} \cup C(x) \cup \left(\bigcup_{y \in C(x)} N(y) \right)$$

is a self-determined set.

7.2. Bethe Lattice Bounds

For site percolation on a Bethe lattice with the coordination number (i.e., number of neighbors of each site) M, one gets

$$\bar{\chi}(p) = \frac{1+p}{1-Mp} \stackrel{\text{def}}{=} \bar{\chi}_M^{\text{B.L.}}(p) \tag{7.7}$$

which is analogous to (2.18).

By the argument which led to Proposition 2.1 we have its following standard analog.

Proposition 7.1. In a site percolation model, with $N < \infty$ [N defined in (7.1)],

$$\bar{\chi}(p) \leqslant \bar{\chi}_N^{\text{B.L.}}(p) \tag{7.8}$$

and, in particular

$$p_c \geqslant N^{-1} \tag{7.9}$$

For the standard model on $\mathbb{Z}^d$, N may be replaced in (7.8), (7.9) by $N-1 = 2d-1$.

7.3. A Lower Bound for γ

Proposition 7.2. In any homogeneous site percolation model

$$\chi(p) \geqslant \frac{pp_c}{|p_c - p|_+} \tag{7.10}$$

In particular, the critical exponent γ, defined by the analog of (2.14), satisfies

$$\gamma \geqslant 1 \tag{7.11}$$

The proof is a direct adaptation of the proof of Proposition 3.1. By applying Russo's lemma to site percolation we have (for infinite systems only formally)

$$\frac{d}{dp}\tau(0,x) = \sum_{y\in\mathbb{L}} \text{Prob}(y \text{ is pivotal for the connection of } x \text{ with } 0) \tag{7.12}$$

Let $S = C_{\mathbb{L}\setminus\{y\}}(0)$. Then $\bar{S}$ is a self-determined set, and (for $y \neq 0$)

$$\begin{aligned}
&\text{Prob}(y \text{ is pivotal for the connection of } x \text{ with } 0)\\
&= \text{Prob}\big(\bar{S} \ni y \text{ and } \mathscr{L}_{\mathbb{L}\setminus S}(y) \ni x\big)\\
&\leqslant \sigma(0,y)\sigma(y,x) = \frac{1}{p^2}\tau(0,y)\tau(y,x)
\end{aligned} \tag{7.13}$$

(by conditioning on $\bar{S}$).

Substituting (7.13) in (7.12) and summing over x we get

$$\frac{d\chi}{dp} \leqslant \frac{1}{p^2}\chi^2 \tag{7.14}$$

and hence the following analog of (3.3):

$$\left|\frac{d\chi^{-1}}{dp}\right| \leqslant \left|\frac{d}{dp}p^{-1}\right| \tag{7.15}$$

The arguments used in the proof of Proposition 3.1, show how to extract from this bound, whose derivation for infinite systems is only formal, an actual proof of Proposition 7.2.

7.4. Inequalities for the Connectivity Functions

The following results hold for general, i.e., not necessarily homogeneous, site percolation models.

Proposition 7.3. For any $V = \{x_1, \ldots, x_n\} \subset \mathbb{L}$ and $A \subset \mathbb{L}$,

$$\sigma(V) - \sigma_{\mathbb{L}\setminus A}(V) \leqslant \sum_{y \in A} \sum_{\substack{x_1 \in W \subset V \\ |W| < |V|}} \sigma(W \cup \{y\})\sigma((V \setminus W) \cup \{y\}) \tag{7.16}$$

One can prove (7.16) by a direct adaptation of the proof of Proposition 4.2. The main difference is that the condition in (4.11), that y is connected to S by an occupied bond, is replaced by the condition that the site y is occupied and $N(y) \cap S \neq \emptyset$. It is convenient to formulate the proof for the functions τ and only at the end absorb the extra factor p by a change to σ.

By the argument used in the proof of Lemma 4.1, (7.16) has the following implication.

Lemma 7.1.

$$\sigma(\{x_1, \ldots, x_n\}) \leqslant \sum_y \sum_{W \subsetneqq \{x_2, \ldots, x_n\}} \sigma(x, y)\sigma(W \cup \{y\}) \times \sigma((\{x_2, \ldots, x_n\} \setminus W) \cup \{y\}) \tag{7.17}$$

Let us now denote, for $A \subset \mathbb{L}$,

$$\hat{\sigma}^A(x, y) = \text{Prob}(x \text{ and } y \text{ are connected by a path of occupied, neighboring, sites which avoids } A\text{, except possibly at one end point } (x \text{ or } y)) \tag{7.18}$$

The following bound is the analog of the Simon–Lieb-type inequality of

Proposition 5.2. Its proof requires only a minor modification, like the one mentioned above.

Proposition 7.4. For each $x, z \in \mathbb{L}$ and $A \subset \mathbb{L}$

$$\sigma(x,z) - \sigma_{\mathbb{L}\setminus A}(x,z) \leqslant \sum_{y \in A} \hat{\sigma}^A(x,y)\sigma(y,z) \tag{7.19}$$

7.5. Tree Diagram Bounds

Iterating Lemma 7.1 we obtain the following tree diagram bounds for site percolation.

Proposition 7.5. The inequality (4.6), with τ replaced by σ in both the left-hand side and the definition of T (4.3), is valid also for general independent site percolation models. In particular,

$$\sigma(x_1, x_2, x_3) \leqslant \sum_{y \in \mathbb{L}} \sigma(x_1, y)\sigma(x_2, y)\sigma(x_3, y) \tag{7.20}$$

7.6. Exponential Decay

The above result implies that the bounds on the *cluster size distribution*, of Section 5.1, hold for site percolation as well, provided one replaces there $|C(x)|$ by $|\mathscr{L}(x)|$. In particular, we get

Proposition 7.6. If in an independent site percolation model $\chi(= p\bar{\chi})$ is finite, then for every $x \in \mathbb{L}$ and $k \geqslant \bar{\chi}^2$

$$\text{Prob}(|C(x)| \geqslant k) \leqslant \text{Prob}(|\mathscr{L}(x)| \geqslant k) \leqslant (e/k)^{1/2} e^{-k/(2\bar{\chi}^2)} \tag{7.21}$$

Similarly, the results of Section 5.3 on the *exponential decay of the two-point function* apply also to site percolation—with σ and $\bar{\chi}$ replacing τ and χ, and the set of bonds used in the definition of $\rho(x, y)$ defined as

$$\mathscr{B} = \{b = \{x, y\} \mid y \in N(x)\}$$

With the norm $\|x\|$ defined by (5.18) we have the following bound.

Proposition 7.7. In any independent site percolation model, with $\chi = p\bar{\chi} < \infty$

$$\tau(0,x) = p\sigma(0,x) \leqslant p(1 - \bar{\chi}^{-1})^{\|x\|} \leqslant pe^{-p\|x\|/\chi} \tag{7.22}$$

The proof is by the argument of Proposition 5.3. That result required only the Simon–Lieb-type inequality of Proposition 5.2–for which a perfect site percolation analog is found in Proposition 7.4, and Lemma 5.1. The FKG argument used in (5.25) for the proof of Lemma 5.1 applies also to the function σ, as can be seen by employing the *asymmetric* expression of (7.3).

7.7. The ∇ Criterion for the Upper Critical Dimension

We complete this section by noting that the site version of the Proposition 6.1 (= 3.2) is also valid. The proof proceeds as the proof of Proposition 6.1, with (6.4) and (6.5) replaced by formulas like those seen in (7.12) and (7.13).

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